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Computation of Three-Dimensional Transonic Internal Flow in Cylindrical Coordinates

W. J. Phares
ARO, Inc.

August 1981

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ARNOLD ENGINEERING DEVELOPMENT CENTER
ARNOLD AIR FORCE STATION, TENNESSEE
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE

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The Euler equations for the inviscid rotational flow of a perfect gas are solved for three-dimensional subsonic-transonic internal flow by use of a time-dependent numerical technique. The numerical approach, which is an extension and modification of that of Cline for two-dimensional flows, is based on the use of the MacCormack finite-difference method for the interior field points. The reference-plane method of characteristics is used for

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20. ABSTRACT (Continued)

coupling the interior field solution to the boundary points. Both the basic equations and the numerical procedures are described, as is the computer program which was written in FORTRAN IV language for either the Cray-1 or the IBM 370/165 computer. As presently written, the program is applicable to the computation of three-dimensional flow in both axisymmetric and relatively simple three-dimensional nozzle geometries. The validity of the computer program was established by computing, in various ways, an axisymmetric nozzle flow as a three-dimensional flow; the numerical results are in good agreement with the results from a well-established computer program for axisymmetric flow.

PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC). The results presented were obtained by ARO, Inc., AEDC Group (a Sverdrup Corporation Company), operating contractor for the AEDC, AFSC, Arnold Air Force Station, Tennessee. Elton R. Thompson was the Air Force project manager. The work was done under ARO Project No. E32A-01, and the manuscript was submitted for publication October 6, 1980.

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1.0 INTRODUCTION

Theoretical calculations of exhaust nozzle performance are often required to aid in the evaluation of propulsion systems tested at the Arnold Engineering Development Center (AEDC). Computer codes have been available for axisymmetric transonic nozzle flow (Refs. 1 and 2) and the supersonic portion of axisymmetric flow fields can be calculated with the well-known method of characteristics.

Even with axisymmetric nozzle geometries, the exhaust nozzle flow field is often three-dimensional (3-D) because of asymmetries in the entrance flow; future tests at AEDC will involve engines with truly three-dimensional nozzle geometry. Consequently, there is a pressing need for the capability to make computations of three-dimensional nozzle flow fields. A computer program has been developed for calculating the supersonic flow in 3-D nozzles (Ref. 3), but no program has been available for computing the subsonic-transonic flow in such nozzles.

In the present study, a computer program has been developed for the inviscid subsonic-transonic flow in three-dimensional propulsion nozzles. Of course, it is not limited to computation of propulsion nozzle performance; it is also applicable to many other 3-D internal flows. Because of the unavailability of detailed experimental data or other analytical solutions for 3-D nozzles, the current program was evaluated using axisymmetric nozzle geometry with the computational axis offset from the nozzle axis to simulate 3-D flow fields.

2.0 ANALYSIS

2.1 APPROACH

The Eulerian equations in nonconservative form are solved for the three-dimensional, inviscid rotational flow of a perfect gas with a time-dependent numerical technique. The objective of this research was to extend and modify the successful two-dimensional (2-D) Cline method (Ref. 1) to provide a 3-D capability.

According to Cline, long computation times associated with time-dependent 2-D calculations are usually required because inefficient algorithms or poor treatment of boundaries demand excessively fine computational meshes. Cline, using the MacCormack scheme coupled with characteristic boundary conditions, produced a 2-D code with reasonable computational times. His success motivated the present approach to the 3-D problem. An outline of the approach follows:

1. The Eulerian equations in nonconservative form are solved.

2. Interior mesh point properties are computed using the efficient MacCormack finite difference scheme.

3. The inlet and wall boundary mesh point properties are calculated using a reference-plane characteristic technique.

4. Exit mesh point properties are calculated using linear extrapolation for supersonic flow and a characteristics scheme for the subsonic case.

5. Singularities along the z-axis, attributable to the choice of a cylindrical coordinate system, are avoided by excluding the z-axis from the flow field calculations.

6. Accumulation of truncation error is reduced by an alternating scheme for the backward-forward option in MacCormack's method.

Note that items 4 and 5 are different from Cline's approach. The physical (z, r, θ) space is transformed into a right-circular cylinder computational domain (ξ, η, ζ) by a coordinate transformation. The computational mesh is uniform in the ξ, η, and ζ directions.

The physical space mesh has equal spacing in the axial (z) and circumferential (θ) directions, whereas that in the radial (r) directions may be unequal.

The temporal step size, Δt, is controlled by the Courant-Fredricks-Lewy (CFL) condition. In general, no smoothing or damping techniques are required to maintain stability.

2.2 GOVERNING EQUATIONS

The governing equations in cylindrical coordinates for time-dependent inviscid, adiabatic 3-D flow of an ideal gas are

Continuity:

$$\frac{D\rho}{Dt} + \nabla \cdot (\rho \hat{v}) = 0 \tag{1}$$

Momentum:

$$\left[\frac{D v_r}{Dt} - \frac{v_\theta}{r}\right] + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{2}$$

$$\left[\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right] + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0 \tag{3}$$

$$\left[\frac{Dv_z}{Dt}\right] + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{4}$$

Energy:

$$\left[\frac{Dp}{Dt}\right] - a^2\frac{D\rho}{Dt} = 0 \tag{5}$$

State:

$$p = \rho RT \tag{6}$$

where

$$\frac{D(\;)}{Dt} = \frac{\partial(\;)}{\partial t} + v_r\frac{\partial(\;)}{\partial r} + \frac{v_\theta}{r}\frac{\partial(\;)}{\partial \theta} + v_z\frac{\partial(\;)}{\partial z} \tag{7}$$

and

$$\nabla \cdot (\rho\vec{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) \tag{8}$$

and where ρ is the density, v_z is the axial velocity, v_r is the radial velocity, v_θ is the circumferential velocity, p is the pressure, a is the local speed of sound, t is the time, and z, r, θ are the axial, radial, and circumferential coordinates.

The physical space (z, r, θ) is mapped into a right-circular cylinder computation domain (ξ, η, ζ) by the following coordinate transformation:

$$\xi = z \quad , \quad \eta = \frac{r}{r_w} \quad , \quad \zeta = \frac{\theta}{2\pi}$$

where r_w denotes the wall values. In the (ξ, η, ζ) coordinate system, Eqs. (1) through (5) become (where the subscripts outside parentheses indicate partial differentiation):

Continuity:

$$
\begin{aligned}
&(\rho)_t + \overline{V}(\rho)_\eta + \frac{\alpha_2}{r} v_\theta (\rho)_\eta + \frac{\delta v_\theta}{r} (\rho)_\zeta + v_z (\rho)_\xi \\
&+ \beta\rho \left(v_r\right)_\eta + \frac{\alpha_2 \rho}{r} \left(v_\theta\right)_\eta + \frac{\delta}{r} \rho \left(v_\theta\right)_\zeta \\
&+ \rho (v_z)_\xi - \alpha_1 \rho \left(v_z\right)_\eta + \rho \frac{v_r}{r} = 0
\end{aligned}
\tag{9}
$$

Momentum:

$$
\begin{aligned}
&\left(v_r\right)_t + \overline{V}\left(v_r\right)_\eta + v_z \left(v_r\right)_\xi + \delta \frac{v_\theta}{r} \left(v_r\right)_\zeta + \frac{\beta}{\rho} (p)_\eta \\
&- \frac{v_\theta^2}{r} + \frac{\alpha_2 v_\theta}{r} \left(v_r\right)_\eta = 0
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\left(v_\theta\right)_t + \overline{V}\left(v_\theta\right)_\eta + \frac{\alpha_2 v_\theta}{r} \left(v_\theta\right)_\eta + \frac{\delta v_\theta}{r} \left(v_\theta\right)_\zeta + v_z \left(v_\theta\right)_\xi \\
&+ \frac{v_r v_\theta}{r} + \frac{\alpha_2}{\rho r} (p)_\eta + \frac{\delta}{\rho r} (p)_\zeta = 0
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\left(v_z\right)_t + \overline{V}\left(v_z\right)_\eta + \frac{\alpha_2 v_\theta}{r} \left(v_z\right)_\eta + \frac{\delta v_\theta}{r} \left(v_z\right)_\zeta + v_z \left(v_z\right)_\xi \\
&+ \frac{1}{\rho} (p)_\xi + \frac{\alpha_1}{\rho} (p)_\eta = 0
\end{aligned}
\tag{12}
$$

Energy:

$$
\begin{aligned}
&(p)_t + \overline{V}(p)_\eta + \frac{\alpha_2}{r} v_\theta (r)_\eta + \frac{\delta}{r} v_\theta (p)_\zeta + v_z (p)_\xi \\
&- a^2 \left[(\rho)_t + \overline{V}(\rho)_\eta + \frac{\alpha_2 v_\theta}{r} (\rho)_\eta + \frac{\delta v_\theta}{r} (\rho)_\zeta \right. \\
&\left. + v_z (\rho)_\xi \right] = 0
\end{aligned}
\tag{13}
$$

where

$$\overline{V} = \beta v_r + \alpha_1 v_z$$

$$\beta = (\eta)_r = \frac{1}{r_w}$$

$$\alpha_1 = (\eta)_z = -\beta\eta\left(r_w\right)_z$$

$$\alpha_2 = (\eta)_\theta = -\beta\eta\left(r_w\right)_\theta$$

and

$$(\zeta)_z = (\zeta)_r = 0 \quad , \quad \delta = (\zeta)_\theta = \frac{1}{2\pi}$$

3.0 NUMERICAL METHOD

The computational space is divided into four types of mesh points: interior, inlet, exit, and wall.

3.1 INTERIOR MESH POINTS

The interior mesh point values are calculated by the second-order accurate, explicit MacCormack finite-difference method, using an alternating backward-forward scheme. The alternating backward-forward scheme used is illustrated in Fig. 1. The governing equations are left in nonconservative form.

3.2 INLET MESH POINTS

The inlet mesh points for subsonic flow are computed using a second-order, reference plane characteristic scheme. In this method, the partial derivatives with respect to η, ζ are computed in the initial value and solution surfaces using noncentered differences as in the MacCormack scheme. These approximations are then transferred to the right-hand side (RHS) in the governing equations, and the resulting system of equations is solved in the constant η, ζ reference planes using a two-independent-variable characteristics scheme.

The characteristics relations derived in Appendix A that relate the interior flow to the nozzle inlet flow are

$$dp - \rho a\, dv_z = \left(a^2 \psi_1 - \rho a \psi_4 + \psi_5\right)dt$$

$$\text{for} \quad \frac{d\xi}{dt} = v_z - a \tag{14}$$

where ψ is defined in Appendix A.

$\partial/\partial z$ ~	1 2 B F	1 2 F B	1 2 B F	1 2 F B	1 2 B F	1 2 F B	1 2 B F	1 2 F B
Time Step ~	1	2	3	4	5	6	7	8

$\partial/\partial r$ ~	1 2 B F	1 2 B F	1 2 F B	1 2 F B	1 2 B F	1 2 B F	1 2 F B	1 2 F B
Time Step ~	1	2	3	4	5	6	7	8

$\partial/\partial \theta$ ~	1 2 B F	1 2 B F	1 2 B F	1 2 B F	1 2 F B	1 2 F B	1 2 F B	1 2 F B
Time Step ~	1	2	3	4	5	6	7	8

1 - First Step of Two-Step Method
2 - Second Step of Two-Step Method
B - Backward Difference
F - Forward Difference

Figure 1. Alternating scheme for the backward-forward option in MacCormack's method.

Use of a reference-plane characteristics scheme requires the specification of the inlet flow angles as well as the stagnation conditions. The equations relating the total and static conditions are

$$p_T/p = \left[1 + (\gamma - 1)\,M^2/2\right]^{\gamma/(\gamma-1)} \tag{15}$$

$$T_T/T = 1 + (\gamma - 1)\,M^2/2 \tag{16}$$

where γ is the ratio of specific heats, M is the Mach number, T is the temperature, and the subscript T denotes the total (stagnation) condition.

Equation (14) is solved using standard characteristic techniques, with the ψ terms evaluated in the initial value plane. Equations (14), (15), and (16), along with the inlet flow angles and the equation of state, form a system of five equations for the five dependent variables.

3.3 EXIT MESH POINTS

For subsonic exit flow, a reference-plane characteristic scheme similar to the inlet scheme is used. The exit pressure is specified. The characteristic relations relating the interior flow to the nozzle exit flow, derived in Appendix A, are

$$\left.\begin{aligned} d\,v_r &= \psi_2\,dt \quad &(17)\\ d\,v_\theta &= \psi_3\,dt \quad &(18)\\ dp - a^2\,d\rho &= \psi_5\,dt \quad &(19)\end{aligned}\right\} \text{ for } \frac{d\xi}{dt} = v_z$$

and

$$dp + \rho a\,d\,v_z = \left(a^2\,\psi_1 - \rho a\,\psi_4 + \psi_5\right)dt$$

$$\text{for} \quad \frac{d\xi}{dt} = v_z + a \tag{20}$$

Equations (17), (18), (19), and (20), along with the exit pressure condition, form a system of five equations for the dependent variables.

For supersonic flow, the flow conditions at the exit mesh points are computed by linear extrapolation.

3.4 WALL MESH POINTS

The wall mesh points are also computed using a reference-plane characteristic scheme. In this scheme, the derivatives with respect to ξ, ζ are approximated and the resulting system of equations is solved in the ξ, ζ = constant reference planes.

The characteristic relations derived in Appendix B which relate the interior flow to the flow at the nozzle walls are

$$\left.\begin{aligned} dp - a^2\,d\rho &= \psi_5\,dt \quad &(21)\\ dv_r - \frac{\beta r}{\alpha_2}\,dv_\theta &= \left(-\frac{\beta r}{\alpha_2}\,\psi_3 + \frac{\psi_2}{r}\right)dt \quad &(22)\\ dv_r - \frac{\beta}{\alpha_1}\,dv_z &= \left(\psi_2 - \frac{\beta}{\alpha_1}\,\psi_4\right)dt \quad &(23)\end{aligned}\right\} \text{ for } \frac{d\eta}{dt} = \overline{V}$$

$$dp + \rho a \frac{\alpha_1}{\alpha^*} dv_z + \rho a \frac{\beta}{\alpha^*} dv_r + \rho a \frac{\alpha_2}{\alpha^*} dv_\theta$$

$$= \left(\psi_5 + a^2 \psi_1 + \rho a \frac{\beta}{\alpha^*} \psi_2 + \frac{\rho a}{r} \frac{\alpha 2}{\alpha^*} \psi_3 + \rho a \frac{\alpha 1}{\alpha^*} \psi_4 \right) dt$$

$$\text{for} \quad \frac{d\eta}{dt} = \overline{V} + a\alpha^* \tag{24}$$

where

$$\alpha^* = \left(\beta^2 + \frac{\alpha_2^2}{r} + \alpha_1^2 \right)^{1/2}$$

and

$$\overline{V} = \beta v_r + \frac{\alpha_2 v_\theta}{r} + \alpha_1 v_z$$

Equations (21), (22), (23), and (24), along with the wall boundary condition, form a system of five equations for the dependent variables. The wall boundary condition is given by

$$v_z F_z + v_r F_r + \frac{v_\theta}{r} F_\theta = 0$$

where F = constant defines the wall suface and F_z, F_r, and F_θ are the corresponding partial derivatives.

3.5 TIME STEP SIZE

The time step size, Δt, is controlled by the well-known CFL condition which can be expressed as

$$\Delta t \le A \Big/ \left\{ (V + a) \left[1/(\Delta\eta/\beta)^2 - 1/(\Delta\xi)^2 + 1/(\Delta\zeta)^2 \right] \right\} \tag{25}$$

where V is the velocity magnitude. Experience gained in the present study indicates that values of A from 1.0 to 2.0 are satisfactory.

4.0 RESULTS AND DISCUSSION

Several adiabatic flow cases were selected to verify the numerical approach described in the preceding sections. The test cases were:

1. Steady-state uniform flow in a cylindrical duct with an initial disturbance at one mesh point.

2. Axisymmetric steady flow in a choked converging-diverging nozzle. The nozzle geometry was the same as for Cline's Case 1 (Ref. 2) and consisted of a 45-deg conical inlet section, a circular arc throat section, and a 15-deg conical diverging section (Fig. 2).

3. Case 2 with a nonaxisymmetric swirl induced by a "bump" in the otherwise uniform inlet total pressure distribution.

4. Case 2 with the computational axis offset various amounts from the nozzle axis, which simulates complex 3-D flow fields.

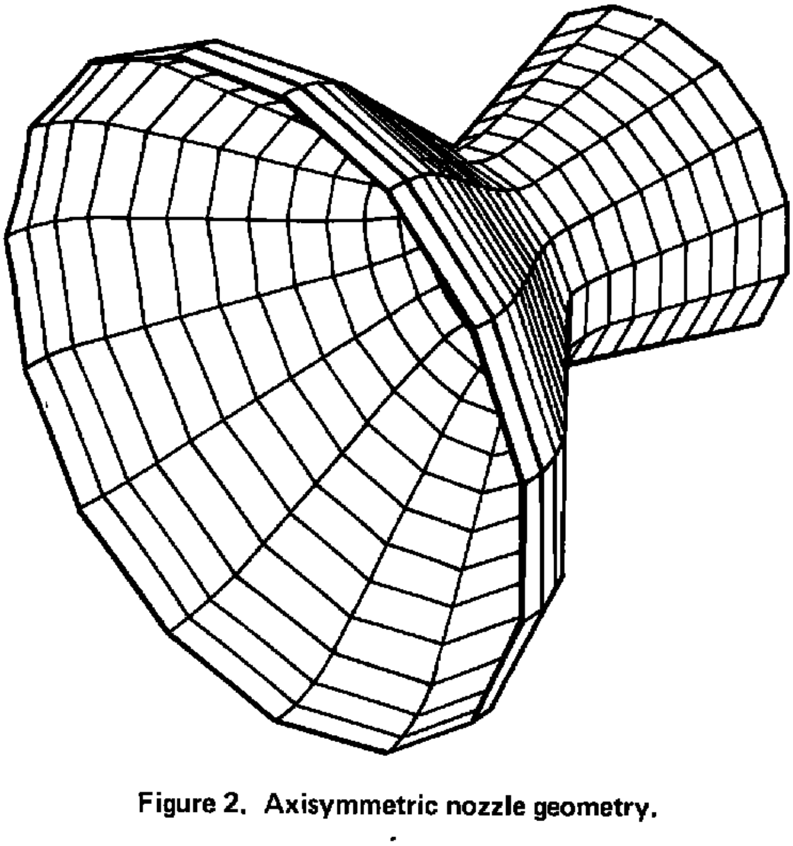


Figure 2. Axisymmetric nozzle geometry.

Cases 1 and 2 were used to debug and verify the computer program for rather simple flows with known steady-state solutions. For Case 1, the solution relaxed smoothly toward uniform flow. (In none of the test cases did the solution at very large times tend to diverge from the known steady solution). The steady-state solution for Case 2 was negligibly different from the results reported by Cline.

A converged steady solution obtained for Case 3 indicates that the numerical approach is applicable to predicting the effect of nonsymmetrical inlet profiles on the flow in axisymmetric nozzles.

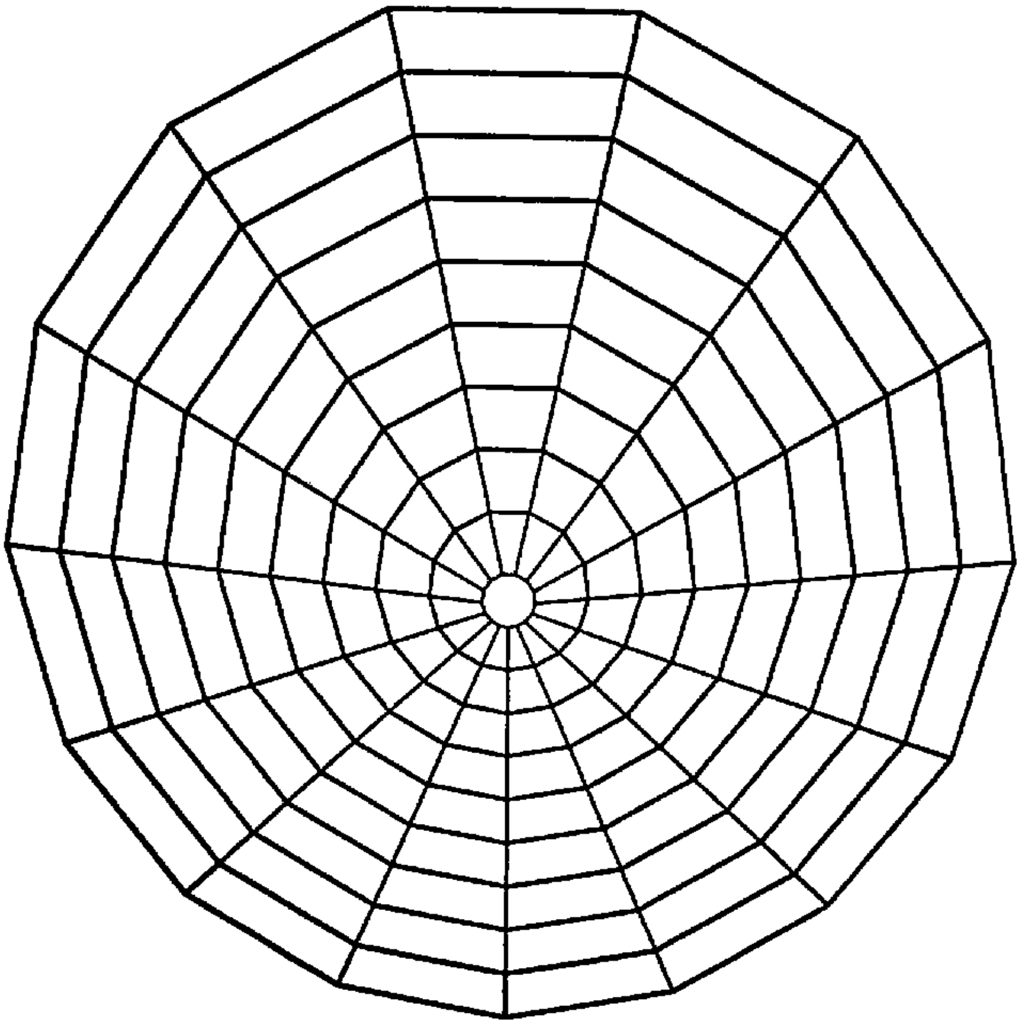
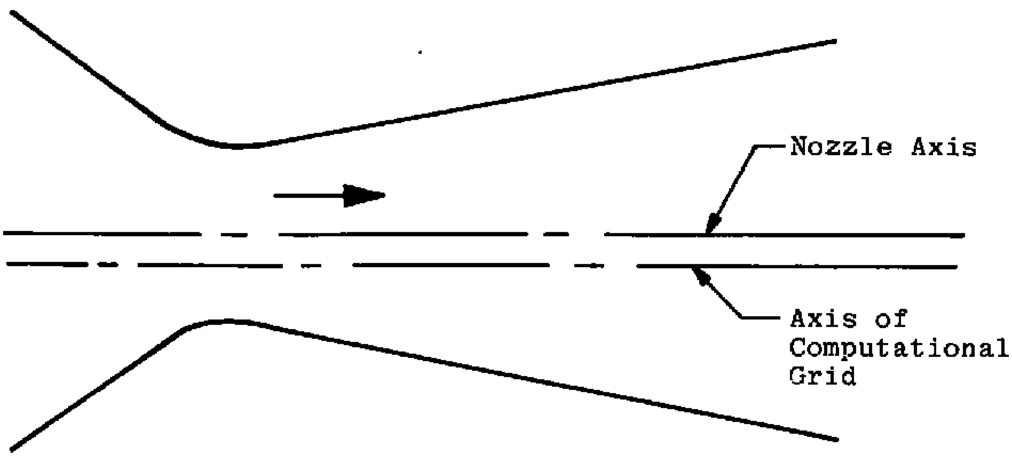


Figure 3. Computational mesh in throat plane for case 4 (20-percent offset).

For Case 4, the computational axis was offset from the axisymmetric nozzle axis by 6, 10, and 20 percent of the nozzle throat radius, r*. The computational mesh in the throat plane is illustrated in Fig. 3 for 20-percent offset. The meshes used and the run times on the Cray-1 computer are summarized in Fig. 4 for the three axis offsets. For all three offsets, the steady-state static pressures throughout the flow field differed less than one percent from those calculated with the axisymmetric Cline program (which correlate well with experiment). These computations of axisymmetric nozzle flow, done the "hard way" with the 3-D program, indicate that the program is acceptably accurate and is capable of computing 3-D subsonic-transonic flow.

With one exception, all solutions were obtained without application of smoothing or damping techniques to maintain stability. The exception was Case 4c, with 20-percent axis offset; in that one case, stability could not be achieved without a modification of MacCormack's scheme which adds a second order truncation error and has the effect of an artificial viscosity. Whether the use of such damping techniques can be avoided by proper mesh selection remains a subject for further investigation.



Case	Offset/r*	Grid	Run Time, min (Cray)
4a	0.06	21 x 8 x 8	0.37
4b	0.10	21 x 12 x 12	3.70
4c	0.20	21 x 12 x 18	10.1

Figure 4. Axisymmetric nozzle with offset computational axis.

5.0 CONCLUDING REMARKS

The Cline numerical approach, which is applicable to planar and axisymmetric nozzle flows, has been successfully extended to three-dimensional flows in this study. Results obtained for the various test flows indicate that the present computer program is applicable to prediction of the effect of nonsymmetric inlet profiles on the flow in axisymmetric nozzle geometries. In addition, the computer program seems well-suited to the computation of the flow fields in relatively simple 3-D nozzle geometries. However, additional work remains to be done in extending the program to arbitrary and complex 3-D geometries.

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APPENDIX A
CHARACTERISTIC RELATIONS FOR INLET AND EXIT MESH POINTS

Governing Equations

The governing Eqs. (18) through (22) can be written as

$$(\rho)_t + \overline{V}\,(\rho)_\eta + \beta\rho\left(v_r\right)_\eta + \frac{\alpha_2\,\rho}{r}\left(v_\theta\right)_\eta + \alpha_1\,\rho\left(v_z\right)_\eta =$$

$$-\left\{\frac{\delta\, v_\theta}{r}\,(\rho)_\zeta + v_z(\rho)_\xi + \frac{\delta}{r}\rho\left(v_\theta\right)_\zeta + \rho\left(v_z\right)_\xi + \frac{\rho\, v_r}{r}\right\} \tag{A-1}$$

$$\left(v_r\right)_t + \overline{V}\left(v_r\right)_\eta + \frac{\beta}{\rho}\,(p)_\eta = -\left\{v_z\left(v_r\right)_\xi + \delta\,\frac{v_\theta}{r}\,(v_r)_\zeta - \frac{v_\theta^2}{r}\right\} \tag{A-2}$$

$$\left(v_\theta\right)_t + \overline{V}\left(v_\theta\right)_\eta + \frac{\alpha_2}{\rho r}\,(p)_\eta = -\left\{\frac{\delta\, v_\theta}{r}\left(v_\theta\right)_\zeta + v_z\left(v_\theta\right)_\xi + \frac{v_r\, v_\theta}{r} + \frac{\delta}{\rho r}\,(p)_\zeta\right\} \tag{A-3}$$

$$\left(v_z\right)_t + \overline{V}\left(v_z\right)_\eta + \frac{\alpha_1}{\rho}\,(p)_\eta = -\left\{\frac{\delta\, v_\theta\left(v_z\right)_\zeta}{r} + v_z\left(v_z\right)_\xi + \frac{1}{\rho}\,(p)_\xi\right\} \tag{A-4}$$

$$(p)_t + \overline{V}\,(p)_\eta - a^2\left((\rho)_t + \overline{V}(\rho)_\eta\right) = -\left\{\frac{\delta}{r}\,v_\theta\,(p)_\zeta + v_z\,(p)_\xi - a^2\left[\frac{\delta\, v_\theta}{r}\,(p)_\zeta + v_z\,(\rho)_\xi\right]\right\} \tag{A-5}$$

where $\overline{V} = \beta v_r + \alpha_2\, v_\theta/r + \alpha_1\, v_z$ is the tangential condition

Defining

ψ_1 = RHS of (A-1)

ψ_2 = RHS of (A-2)

ψ_3 = RHS of (A-3)

ψ_4 = RHS of (A-4)

ψ_5 = RHS of (A-5)

then the system of governing Equations (A-1) through (A-5) can be written in vector notation as

$$\tilde{A}\,\vec{W}_t + \tilde{B}\,\vec{W}_\eta = \vec{F} \tag{A-6}$$

where

$$\vec{W} = \begin{vmatrix} \rho \\ v_r \\ v_\theta \\ v_z \\ p \end{vmatrix} \quad \text{and} \quad \vec{F} = \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{vmatrix}$$

and with the A and B coefficient matricies as

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -a^2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} \bar{V} & \beta\rho & \dfrac{\alpha_2\rho}{r} & \alpha_1\rho & 0 \\ 0 & \bar{V} & 0 & 0 & \beta/\rho \\ 0 & 0 & \bar{V} & 0 & \dfrac{\alpha_2}{\rho r} \\ 0 & 0 & 0 & \bar{V} & \dfrac{\alpha_1}{\rho} \\ -a^2\,\bar{V} & 0 & 0 & 0 & \bar{V} \end{bmatrix}$$

Defining Characteristics

The theory of eigenvalues is used to derive the equations of the characteristics. The eigenvalues are defined as

$$\lambda = \frac{d\eta}{dt} \tag{A-7}$$

The total derivative of the $\vec{W}$ vector is

$$\frac{d\vec{W}}{dt} = \vec{W}_t - \frac{d\eta}{dt}\vec{W}_\eta = \vec{W}_t + \lambda \vec{W}_\eta \tag{A-8}$$

which, coupled with the vector equation, Eq. (A-6), yields

$$(\widetilde{B} - \lambda \widetilde{A})\vec{W}_\eta = \vec{F} - \widetilde{A}\frac{d\vec{W}}{dt} \tag{A-9}$$

The eigenvalues and eigenvectors ($\vec{\Lambda}_i$) satisfy the inner product relation, based on Eq. (A-9),

$$\left[(\widetilde{B} - \lambda_i \widetilde{A})\vec{W}_\eta, \vec{\Lambda}_i\right] = \left[\vec{F} - \widetilde{A}\frac{d\vec{W}}{dt}, \Lambda_i\right] = 0 \tag{A-10}$$

or

$$\left[\vec{W}_\eta, (\widetilde{B} - \lambda_i \widetilde{A})^* \vec{\Lambda}_i\right] = \left[\vec{F} - \widetilde{A}\frac{d\vec{W}}{dt}, \Lambda_i\right] = 0 \tag{A-11}$$

where $(\widetilde{B} - \lambda\widetilde{A})^*$ signifies the transpose of the $(\widetilde{B} - \lambda\widetilde{A})$ matrix.

These matrices are

$$(\widetilde{B} - \lambda\widetilde{A}) = \begin{bmatrix} (\bar{V} - \lambda) & \beta\rho & \frac{\alpha_2 \rho}{r} & \alpha_1 \rho & 0 \\ 0 & (\bar{V} - \lambda) & 0 & 0 & \beta/\rho \\ 0 & 0 & (\bar{V} - \lambda) & 0 & \frac{\alpha_2}{\rho r} \\ 0 & 0 & 0 & (\bar{V} - \lambda) & \frac{\alpha_1}{\rho} \\ -a^2(\bar{V} - \lambda) & 0 & 0 & 0 & (\bar{V} - \lambda) \end{bmatrix}$$

and

$$(\widetilde{B} - \lambda\widetilde{A})^* = \begin{bmatrix} (\bar{V} - \lambda) & 0 & 0 & 0 & -a^2(\bar{V} - \lambda) \\ \beta\rho & \bar{V} - \lambda & 0 & 0 & 0 \\ \dfrac{\alpha_2\rho}{r} & 0 & \bar{V} - \lambda & 0 & 0 \\ \alpha_1\rho & 0 & 0 & \bar{V} - \lambda & 0 \\ 0 & \dfrac{\beta}{\rho} & \dfrac{\alpha_2}{\rho r} & \dfrac{\alpha_1}{\rho} & \bar{V} - \lambda \end{bmatrix}$$

The eigenvalues are obtained from the determinate relationship

$$|(\widetilde{B} - \lambda\widetilde{A})^*| = (\bar{V} - \lambda)^5 - a^2(\bar{V} - \lambda)\left[\beta^2(\bar{V} - \lambda)^2 + (\bar{V} - \lambda)\left\{\frac{\alpha_2^2}{r^2}(\bar{V} - \lambda) + (\bar{V} - \lambda)\,\alpha_1^2\right\}\right] = 0 \tag{A-12}$$

The resulting eigenvalue equations, which describe the characteristics, are

$$(\bar{V} - \lambda)^3 = 0 \tag{A-13}$$

and

$$(\bar{V} - \lambda^2) - a^2\left(\beta^2 + \frac{\alpha_2^2}{r^2} + \alpha_1^2\right) = 0 \tag{A-14}$$

Equation (A-13) is easily solved for λ_1, λ_2 and λ_3 characteristics,

$$\lambda_1 = \lambda_2 = \lambda_3 = \bar{V} \tag{A-15}$$

Equation (A-14) is solved for λ_4 and λ_5 characteristics,

$$\lambda_4 = \lambda_5 = \bar{V} \pm a\,\alpha^* \tag{A-16}$$

where

$$\alpha^* = \left(\beta^2 + \frac{\alpha_2^2}{r^2} + \alpha_1^2\right)^{1/2}$$

Compatibility Relations

To derive the compatibility relations along the characteristics, the eigenvectors which define the characteristics must satisfy

$$\left[\left(B - \lambda_i A\right), \vec{\Lambda}_i\right] = 0 \tag{A-17}$$

for $\lambda_{1,2,3} = \overline{V}$, or

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \beta/\rho & 0 & 0 & 0 & 0 \\ \dfrac{\alpha_2 \rho}{r} & 0 & 0 & 0 & 0 \\ \alpha_1 \rho & 0 & 0 & 0 & 0 \\ 0 & \beta/\rho & \dfrac{\alpha_2}{\rho r} & \dfrac{\alpha_1}{\rho} & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \Lambda_i \\ \\ \\ \end{bmatrix} = \Phi$$

The resulting eigenvectors are

$$\vec{\Lambda}_{1,\,2,\,3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha_2 r \\ -\beta \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha_1 \\ 0 \\ -\beta \\ 0 \end{bmatrix} \tag{A-18}$$

also, for $\lambda_{4,5} = \overline{V} \pm a\alpha^*$

$$\begin{bmatrix} -(\pm a\alpha^*) & 0 & 0 & 0 & a^2(\pm a\alpha^*) \\ \beta\rho & -(\pm a\alpha^*) & 0 & 0 & 0 \\ \alpha_2 \rho & 0 & -(\pm a\alpha^*) & 0 & 0 \\ \alpha_1 \rho & 0 & 0 & -(\pm a\alpha^*) & 0 \\ 0 & \beta & \dfrac{\alpha_2}{r} & \alpha_1 & -\rho(\pm a\alpha^*) \end{bmatrix} \begin{bmatrix} \\ \\ \Lambda_i \\ \\ \\ \end{bmatrix} = \Phi$$

the eigenvectors are

$$\Lambda_{4,5} = \begin{bmatrix} -(\pm a\alpha^*)^4 \\ -\beta\rho\,(\pm a\alpha^*)^3 \\ -\frac{\alpha_2\rho}{r}(\pm a\alpha^*)^3 \\ -\alpha_1\rho\,(\pm a\alpha^*)^3 \\ -(\pm a\alpha^*)^2\,\alpha^{*2} \end{bmatrix} \tag{A-19}$$

The compatibility relations along the characteristics are, using the eigenvectors and Eq. (A-11),

$$\vec{F} - \widetilde{A}\frac{d\vec{w}}{dt} = \begin{bmatrix} \psi_1 - \frac{d\rho}{dt} \\ \psi_2 - \frac{dv_r}{dt} \\ \psi_3 - \frac{dv_\theta}{dt} \\ \psi_4 - \frac{dv_z}{dt} \\ \psi_5 + a^2\frac{d\rho}{dt} - \frac{dp}{dt} \end{bmatrix}$$

for

$$\Lambda_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \psi_5 + a^2\frac{d\rho}{dt} - \frac{dp}{dt} = 0$$

and

$$\Lambda_2 = \begin{bmatrix} 0 \\ \alpha_2/r \\ -\beta \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \frac{\alpha_2}{r}\left(\psi_2 - \frac{dv_r}{dt}\right) - \beta\left(\psi_3 - \frac{dv_\theta}{dt}\right) = 0$$

and

$$\Lambda_3 = \begin{bmatrix} 0 \\ \alpha_1 \\ 0 \\ -\beta \\ 0 \end{bmatrix} \quad ; \quad \alpha_1\left(\psi_2 - \frac{dv_r}{dt}\right) - \beta\left(\psi_4 - \frac{dv_z}{dt}\right) = 0$$

and

$$\begin{bmatrix} -(\pm a\alpha^*)^4 \\ -\beta\rho(\pm a\alpha^*)^3 \\ -\frac{\alpha_2\rho}{r}(\pm a\alpha^*)^3 \\ -\alpha_1\rho(\pm a\alpha^*)^3 \\ -(\pm a\alpha^*)^2\,\alpha^{*2} \end{bmatrix}$$

$$-\left(\psi_1 - \frac{d\rho}{dt}\right)(\pm a^*\alpha^*)^4 - \left(\psi_2 - \frac{dv_r}{dt}\right)\beta\rho(\pm a\alpha^*)^3 - \left(\psi_3 - \frac{dv_\theta}{dt}\right)\frac{\alpha_2\rho}{r}(\pm a\alpha^*)^3$$

$$-\left(\psi_4 - \frac{dv_z}{dt}\right)\alpha_1\rho(\pm a\alpha^*)^3 - \left(\psi_5 + a^2\frac{d\rho}{dt} - \frac{dp}{dt}\right)(\pm a\alpha^*)^2\,\alpha^{*2} = 0$$

or

$$dp \pm \rho \alpha_1 \, a dv_z / \alpha^* \pm \rho \beta a dv_r / \alpha^* \pm \rho \frac{\alpha_2}{r} a \frac{dv_\theta}{\alpha^*} = \Big(\psi_5 \pm a^2 \psi_1$$

$$\pm \rho \beta \, a \, \psi_2 / \alpha^* \pm \frac{\alpha_2 \rho a}{r} \frac{\psi_3}{\alpha^*} \pm \alpha_1 \rho a \, \psi_4 / \alpha^* \Big) dt$$

Compatibility relations for (ξ, ζ) constant reference plane are

$$\left.\begin{aligned} & dp - a^2 d\rho = \psi_5 \, dt \\ & dv_r - \frac{\beta r}{\alpha_2} dv_\theta = \left(- \frac{\beta r}{\alpha_2} \psi_3 + \frac{\psi_2}{r} \right) dt \\ & dv_r - \frac{\beta}{\alpha_1} dv_z = \left(\psi_2 - \frac{\beta}{\alpha_1} \psi_4 \right) dt \end{aligned} \right\} , \text{ for } \frac{d\eta}{dt} = \bar{V}$$

and

$$\left.\begin{aligned} & dp \pm \rho \alpha_1 a \frac{dv_z}{\alpha^*} \pm \rho \beta a \frac{dv_r}{\alpha^*} \pm \rho \alpha_2 a \frac{dv_\theta}{\alpha^*} \\ & = \left(\psi_5 \pm a^2 \psi_1 \pm \rho \beta a \, \psi_2 / \alpha^* \pm \frac{\alpha_2}{r} \rho a \, \psi_3 / \alpha^* \pm \alpha_1 \rho a \, \psi_4 / \alpha^* \right) dt \end{aligned} \right\} , \text{ for } \frac{d\eta}{dt} = \bar{V} \pm a \alpha^*$$

APPENDIX B
CHARACTERISTIC RELATIONS FOR WALL MESH POINTS

Governing Equations

The governing Eqs. (18) through (32) can be written as

$$(\rho)_t + \rho\left(v_z\right)_\xi + v_z(\rho)_\xi = -\left\{\bar{V}(\rho)_\eta + \frac{\alpha_2 v_\theta}{r}(\rho)_\eta + \frac{\delta v_\theta}{r}(\rho)_\zeta\right.$$

$$\left. + \beta\rho\left(v_r\right)_\eta + \frac{\alpha_2 \rho}{r}\left(v_\theta\right)_\eta + \frac{\delta}{r}\rho\left(v_\theta\right)_\zeta + \alpha_1\rho\left(v_z\right)_\eta + \frac{\rho v_r}{r}\right\} \quad \text{(B-1)}$$

$$\left(v_r\right)_t + v_z\left(v_r\right)_\xi = -\left\{\bar{V}\left(v_r\right)_\eta + \frac{\delta v_\theta}{r}\left(v_r\right)_\zeta + \frac{\beta}{\rho}(P)_\eta - \frac{v_\theta^2}{r} + \frac{\alpha_2 v_\theta\left(v_r\right)_\eta}{r}\right\} \quad \text{(B-2)}$$

$$\left(v_\theta\right)_t + v_z\left(v_\theta\right)_\xi = -\left\{\bar{V}\left(v_\theta\right)_\eta + \frac{\alpha_2 v_\theta}{r}\left(v_\theta\right)_\eta + \frac{\delta v_\theta}{r}\left(v_\theta\right)_\zeta + \frac{v_r v_\theta}{r}\right\}$$

$$- \frac{\alpha_2}{\rho r}(P)_\eta - \frac{\delta}{\rho r}(P)_\zeta \quad \text{(B-3)}$$

$$\left(v_z\right)_t + v_z\left(v_z\right)_\xi + \frac{1}{\rho}(P)_\xi = -\left\{\bar{V}\left(v_z\right)_\eta + \frac{\alpha_2 v_\theta}{r}\left(v_z\right)_\eta + \frac{\delta v_\theta\left(v_z\right)_\zeta}{r}\right.$$

$$\left. + \frac{\alpha_1}{\rho}(P)_\eta\right\} \quad \text{(B-4)}$$

$$(P)_t - a^2 v_z(\rho)_\xi + v_z(P)_\xi - a^2(\rho)_t = -\left\{\bar{V}(P)_\eta + \frac{\alpha_2}{r} v_\theta (P)_\eta\right.$$

$$\left. + \frac{\delta}{r} v_\theta (P)_\zeta - a^2\left(\bar{V}(\rho)_\eta + \frac{\alpha_2 v_\theta}{r}(\rho)_\eta + \frac{\delta v_\theta}{r}(\rho)_\zeta\right)\right\} \quad \text{(B-5)}$$

where

$$\bar{V} = \beta v_r + \alpha_1 v_z$$

and

$$\psi_1 = \text{RHS of (B-1)}$$

$$\psi_2 = \text{RHS of (B-2)}$$

$$\psi_3 = \text{RHS of (B-3)}$$

$$\psi_4 = \text{RHS of (B-4)}$$

$$\psi_5 = \text{RHS of (B-5)}$$

The theory of eigenvalues, defined in Appendix A, is used to derive the equations of the characteristics and compatibility relations.

Equations (B-1) through (B-5) in vector notation are

$$\widetilde{A}\,\vec{W}_t + \widetilde{B}\,W_\xi = \vec{F}$$

where

$$\vec{W} = \begin{bmatrix} \rho \\ v_r \\ v_\theta \\ v_z \\ P \end{bmatrix} \quad \text{and} \quad \vec{F} = \begin{bmatrix} \psi_2 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

and

$$\widetilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -a^2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
\widetilde{B} = \begin{bmatrix} v_z & 0 & 0 & \rho & 0 \\ 0 & v_z & 0 & 0 & 0 \\ 0 & 0 & v_z & 0 & 0 \\ 0 & 0 & 0 & v_z & \frac{1}{\rho} \\ -a^2 v_z & 0 & 0 & 0 & v_z \end{bmatrix}
$$

then

$$
(\widetilde{B} - \lambda\widetilde{A}) = \begin{bmatrix} (v_z - \lambda) & 0 & 0 & \rho & 0 \\ 0 & (v_z - \lambda) & 0 & 0 & 0 \\ 0 & 0 & (v_z - \lambda) & 0 & 0 \\ 0 & 0 & 0 & (v_z - \lambda) & \frac{1}{\rho} \\ -a^2(v_z - \lambda) & 0 & 0 & 0 & (v_z - \lambda) \end{bmatrix}
$$

$$
(\widetilde{B} - \lambda\widetilde{A})^* = \begin{bmatrix} (v_z - \lambda) & 0 & 0 & 0 & -a^2(v_z - \lambda) \\ 0 & (v_z - \lambda) & 0 & 0 & 0 \\ 0 & 0 & (v_z - \lambda) & 0 & 0 \\ \rho & 0 & 0 & (v_z - \lambda) & 0 \\ 0 & 0 & 0 & \frac{1}{\rho} & (v_z - \lambda) \end{bmatrix}
$$

$$
|(\widetilde{B} - \lambda\widetilde{A})^*| = (v_z - \lambda)^5 - a^2(v_z - \lambda)^3 = (v_z - \lambda)^3 \left((v_z - \lambda)^2 - a^2 \right) = 0
$$

$$
\lambda_1 = \lambda_2 = \lambda_3 = v_z \quad , \quad \lambda_4 = \lambda_5 = v_z \pm a
$$

The eigenvectors and eigenvalues satisfy

$$\left[(\widetilde{B} - \lambda\widetilde{A})^{*} , \Lambda_i\right] = 0$$

for $\lambda_1 = \lambda_2 = \lambda_3 = v_z$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\rho & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \\ \Lambda_5 \end{bmatrix} = \Phi$$

then

$$\vec{\Lambda}_{1,\,2,\,3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_{4,5} = v_r \pm a$

$$\begin{bmatrix} \mp a & 0 & 0 & 0 & \pm a^3 \\ 0 & \mp a & 0 & 0 & 0 \\ 0 & 0 & \mp a & 0 & 0 \\ 0 & 0 & 0 & \mp a & 0 \\ 0 & 0 & 0 & \frac{1}{\rho} & \mp a \end{bmatrix} \begin{bmatrix} \\ \\ \Lambda_i \\ \\ \\ \end{bmatrix} = \vec{\Phi}$$

results

$$\mp a\,\Lambda_1 \pm a^3\,\Lambda_5 = 0 \qquad \Lambda_1 = \pm a^2\,\Lambda_5$$

$$\rho\,\Lambda_1 \mp a\,\Lambda_4 = 0 \qquad \Lambda_1 = \pm \frac{a}{\rho}\,\Lambda_4$$

$$\frac{1}{\rho}\,\Lambda_4 \mp a\,\Lambda_5 = 0 \qquad \Lambda_4 = \pm a\,\rho\,\Lambda_5$$

let $\Lambda_5 = 1$

Then $\Lambda_4 = \pm\ \rho a$, $\Lambda_1 = a^2$

and

$$\vec{\Lambda}_4 = \vec{\Lambda}_5 = \begin{bmatrix} a^2 \\ 0 \\ 0 \\ \pm\,\rho a \\ 1 \end{bmatrix}$$

The compatibility relations along the characteristics are, using the eigenvectors

$$\left[\left(\vec{F} - \widetilde{A}\,\frac{d\vec{W}}{dt}\right), \vec{\Lambda}\right] = 0$$

$$\vec{F} - \widetilde{A}\,\frac{d\vec{W}}{dt} = \begin{bmatrix} \psi_1 - \dfrac{d\rho}{dt} \\ \psi_2 - \dfrac{dv_r}{dt} \\ \psi_3 - \dfrac{dv_\theta}{dt} \\ \psi_4 - \dfrac{dv_z}{dt} \\ \psi_5 + a^2\,\dfrac{d\rho}{dt} - \dfrac{dP}{dt} \end{bmatrix}$$

for

$$\vec{\Lambda}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \quad \psi_2 - \frac{dv_r}{dt} = 0$$

$$\vec{\Lambda}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \psi_3 - \frac{dv_\theta}{dt} = 0$$

$$\vec{\Lambda}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} ; \quad \psi_5 + a^2 \frac{d\rho}{dt} - \frac{dP}{dt} = 0$$

$$\Lambda_{4,5} = \begin{bmatrix} a^2 \\ 0 \\ 0 \\ \mp \rho a \\ 1 \end{bmatrix} ; \quad a^2 \left(\psi_1 - \frac{d\rho}{dt} \right) \pm \rho a \left(\psi_4 - \frac{dv_z}{dt} \right) + \psi_5 + a^2 \frac{d\rho}{dt} - \frac{dP}{dt} = 0$$

Compatibility relations for (η,ξ) constant reference planes are

$$\left.\begin{aligned} dv_r &= \psi_2\,dt \\ dv_\theta &= \psi_3\,dt \\ dP - a^2\,d\rho &= \psi_5\,dt \end{aligned}\right\} \quad \text{for, } \frac{d\xi}{dt} = v_z$$

$$\left. dP \pm \rho a\, dv_z = \left(a^2\,\psi_1 \pm \rho a\,\psi_4 + \psi_5\right) dt \right\} \qquad \text{for, } \frac{d\xi}{dt} = v_z \pm a$$

NOMENCLATURE

A	Premultiplier for the time step
a	Speed of sound
$\tilde{A}, \tilde{B}$	Coefficient matrices
F_z, F_r, F_θ	Local wall slopes
p	Static pressure
P_T	Total pressure
R	Gas constant
r	Radial coordinate
r_w	Wall radius
r^*	Nozzle throat radius
T	Static temperature
t	Time
T_T	Total temperature
V	Velocity magnitude
v	Circumferential velocity component
v_r	Radial velocity component
v_z	Axial velocity component
$\overline{V}$	Tangential velocity condition
W_t, W_η, W_ξ	Partial derivative dependent variable vectors
$\vec{W}$	Dependent variable vector
z	Axial coordinate
Δt	Time step size
γ	Ratio of specific heats

θ	Circumferential coordinate
$\vec{\Lambda}_i$	Eigenvectors
λ	Eigenvalues
ξ, η, ζ	Transformed spatial coordinates corresponding to (z, r, θ)
ϱ	Density
$\vec{\Phi}$	Null vector
$\partial/\partial z$, $\partial/\partial r$, $\partial/\partial \theta$	Partial derivative operators
*	Transpose
•	Dot product vector operator